Superactivation of Bound Entanglement

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We show that, in a multi-party setting, two non-distillable (bound-entangled) states tensored together can make a distillable state. This is an example of true superadditivity of distillable entanglement. We also show that unlockable bound-entangled states cannot be asymptotically unentangled, providing the first proof that some states are truly bound-entangled in the sense of being both non-distillable and non-separable asymptotically.

The joint state of more than one quantum system cannot always be thought of as a separate state of each system, nor even as a correlated mixture of separate states of each system [1], a situation known as quantum entanglement. Entanglement leads to the most counterintuitive effects in quantum mechanics, including the disturbing idea due to Bell that quantum mechanics is incompatible with local hidden variable theories [2]. Even today new quantum oddities with their basis in entanglement are being found, and the study of entanglement is at the heart of quantum information theory.

A state belonging to parties A, B, C, *etc.* is said to be *inseparable* if it cannot be written in separable form

$$\rho^{ABC\cdots} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \rho_i^C \ldots \tag{1}$$

for any positive probabilities p_i summing to one and set of density matrices $\rho_i^A, \rho_i^B, \rho_i^C \ldots$, where, for example, ρ_i^A operates on the Hilbert space belonging to party A. Notice that the superscripts A, B, C, *etc.* denote the parties by whom the state is shared. We say that a state is *distillable* if some pure entangled state shared by some subset of the parties is obtainable (asymptotically [3]) from it by local operations and classical communication (LOCC) amongst the parties.

It is known that many inseparable quantum mixed states are distillable, while separable states are not [4,5]. More recently it has been shown that some mixed states which are entangled in the sense of being inseparable nevertheless cannot be distilled into any pure entanglement [6,7]. Such states are known as *bound-entangled* states.

It has been an open question whether bound-entangled states, though inseparable, are actually entangled at all in an asymptotic sense. A state ρ is said to be *asymptotically unentangled* [3]) if for any positive ϵ there exists a number of copies N, a number m sublinear in N of EPR pairs shared in some way among the parties, and an LOCC method of constructing from those EPR pairs a state ρ' such that $F(\rho^{\otimes N}, \rho') > 1 - \epsilon$ for some sensible definition of the fidelity F between two density matrices [8]. In this letter we show the first example of a bound entangled state that can be proved not to be asymptotically unentangled. Other examples can be found it [9].

In the bipartite case, bound entanglement may sometimes be useful in a kind of quasi-distillation process known as *activating* the bound entanglement [10] in which a finite number of free-entangled mixed states are distilled with the help of a large number of bound-entangled states. This is not a true distillation of the bound entanglement in that no more pure entanglement is produced than the distillable entanglement of the free-entangled mixed states, the distillable entanglement being defined as the pure entanglement distillable per state from an infinite number of copies of a state.

In the case of more than two parties the bound entanglement can be more truly activated by the presence of free entanglement. In examples given by Cirac, Tarrach and Dür [11–13], and in the equivalent formulation of unlockable bound-entangled states [14], when several parties share certain bound entangled states, and when some subset of the parties get to share pure entanglement then some pure entanglement may be distilled between parties where it would be impossible to obtain any without having shared the bound-entangled state. This is a kind of superadditivity of distillable entanglement, though in the known cases no *more* entanglement is distilled than the pure entanglement that was shared, rather it is in a different place. Later in this letter we will look at unlockable states in much more detail since we will need some of the results about them.

In this letter we present an effect we call superactivation of bound entanglement. It is "super" in the sense of *superadditivity* of distillable entanglement, but without the restrictions of either of the earlier types of activation of bound entanglement. In superactivation two entangled mixed states ρ, ρ' are combined to yield more pure entanglement than the sum of what a set of parties could distill from either ρ or ρ' on their own, even if many copies of ρ or ρ' are shared. In particular, both states in our example are bound-entangled states from which no pure entanglement can be distilled. Our result thus provides the first example of superaddivity of distillable entanglement.

We will use the usual notation for the maximally en-

tangled states of two qubits (the Bell states):

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\right),\ |\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle\right) \quad (2)$$

For convenience we adopt the following notation as well:

$$\Psi = \{\Psi^-, \Psi^+, \Phi^+, \Phi^-\} \text{ with elements } \Psi_i \text{ , and} \quad (3)$$

$$\sigma = \{\mathbf{1_2}, \left(\begin{smallmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{-1}\end{smallmatrix}\right), \left(\begin{smallmatrix}\mathbf{0} & \mathbf{-1}\\ \mathbf{1} & \mathbf{0}\end{smallmatrix}\right), \left(\begin{smallmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{smallmatrix}\right)\} \text{ with elements } \sigma_{\mathbf{i}} \ , \quad (4)$$

where $\mathbf{1_2}$ is the identity matrix in 2×2. In the text, we shall refer to a Bell state as any one of the four states (3) and to an EPR state as the standard singlet state $|\Psi^-\rangle$. The Bell states $|\Psi_i\rangle$ are related to the standard EPR state $|\Psi^-\rangle$ by the following identities, up to an overall phase which is unimportant here:

$$|\Psi^-\rangle = \mathbf{1_2} \otimes \sigma_{\mathbf{i}}|\mathbf{\Psi_i}\rangle = \sigma_{\mathbf{i}} \otimes \mathbf{1_2}|\mathbf{\Psi_i}\rangle \quad (5)$$

$$|\Psi_i\rangle = \mathbf{1_2} \otimes \sigma_{\mathbf{i}}|\mathbf{\Psi^-}\rangle = \sigma_{\mathbf{i}} \otimes \mathbf{1_2}|\mathbf{\Psi^-}\rangle \ . \quad (6)$$

In teleportation [15], A and B share an EPR pair $|\Psi^-\rangle$, and A has another qubit in a state $|\psi\rangle$. A first does a joint measurement on her two qubits in the basis formed by the Bell states. There are four equally likely outcomes corresponding to the Bell states $|\Psi_i\rangle$. B's half of the EPR pair after this measurement is $\sigma_i|\psi\rangle$ up to a phase that can be ignored. Then A communicates i to B who then performs a rotation σ_i on his state giving $\sigma_i^2|\psi\rangle$. But $\sigma_i^2 = \pm\mathbf{1_2}$ and thus the final state B has is $|\psi\rangle$ up to a phase.

An easy lemma about teleportation is that if a state $|\psi\rangle$ is teleported from A to B using an incorrect one of the Bell states $|\Psi_i\rangle$ rather than $|\Psi^-\rangle$ as normally required by the protocol, then the result of the teleportation will be $\sigma_i|\psi\rangle$, again up to an overall phase. This is easily seen by using (6) to write the incorrect Bell state as $|\Psi^-\rangle$ with a σ_i operating on B's part of the $|\Psi^-\rangle$ i.e. $|\psi\rangle = \mathbf{1_2} \otimes \sigma_{\mathbf{i}}|\mathbf{\Psi^-}\rangle$. If A's outcome of the Bell measurement is j then B's corresponding state is $\sigma_i\sigma_j|\psi\rangle$. Thus after B applies the rotation σ_j the state becomes $\sigma_j\sigma_i\sigma_j|\psi\rangle$. If the rotation σ_j which is the final step in teleportation could be squeezed in before the σ_i the proof would be complete, but instead it follows the σ_i. However, the rotations used in teleportation are also the σ matrices, and all the σ_i, σ_j either commute or anticommute ($\sigma_i\sigma_j = \pm\sigma_j\sigma_i$) and so their order can be freely interchanged up to a phase. Thus the lemma is proved. □

In [14] a four-party bound entangled state was presented:

$$\rho^{ABCD} = \frac{1}{4}\sum_{i=0}^{3} |\Psi_i\rangle^{AB}\langle\Psi_i| \otimes |\Psi_i\rangle^{CD}\langle\Psi_i| \quad (7)$$

In other words, A and B share one of the four Bell states, but don't know which one, and C and D share the same Bell state, also not knowing which one.

This state has several properties:

• Symmetry under interchange of parties: $\rho^{ABCD} = \rho^{ABDC} = \rho^{ADBC}$, *etc.* This may be verified by writing out the state as a $16 \otimes 16$ matrix and interchanging indices. A more enlightening way is to use our lemma and think of the state in terms of teleportation. First, we note that some of the symmetries are obvious, for example interchanging A and B because Bell states are themselves symmetric under interchange. So the only symmetry we need to consider is the interchange of B with C and the rest can be constructed trivially.

Consider the state in its original form, with A and B sharing an unknown Bell state and C and D sharing the same one. Now consider A and C getting together and performing a Bell measurement and obtaining the result $|\Psi_j\rangle$, which we can think of as A and C doing the first step required to teleport A's particle to D using the unknown Bell state shared by C and D. The result $|\Psi_j\rangle$ is random since A and C had halves of completely separate unknown Bell states. The state being teleported is half of a Bell state given by Eq. (6) $\sigma_i \otimes \mathbf{1_2}|\mathbf{\Psi^-}\rangle$ as is the state used in the teleportation. So, by our lemma, if the teleportation were completed an extra σ_i would be introduced, and the two σ_i's would cancel being self-inverse (up to a phase). Thus, B and D would share a standard $|\Psi^-\rangle$. But if the σ_i needed to complete teleportation is not performed, this means that B and D share the Bell state $\sigma_j^{-1} \otimes \mathbf{1_2}|\mathbf{\Psi^-}\rangle = \sigma_{\mathbf{j}} \otimes \mathbf{1_2}|\mathbf{\Psi^-}\rangle = |\mathbf{\Psi_j}\rangle$ (ignoring phases), which is the result obtained by A and C. So AC and BD share identical random Bell states, which was the original form of the density matrix, but with A and C interchanged.

• Non-distillability: When all four parties remain separated and cannot perform joint quantum operations, then they cannot distill any pure entanglement by LOCC, even if they share many states, each having density matrix ρ^{ABCD}. This comes from the fact every party is separated from every other across a separable cut. This is easy to see since the state (7) is separable across the $AB : CD$ cut by construction and the state has the symmetry property.

• Unlockability: The entanglement of the state can be *unlocked*. If A and B come together and perform a joint quantum measurement, they can determine which of the four Bell states they have (the four Bell states form an orthogonal basis) and tell C and D the outcome. Since C and D then know which Bell state they share, they can convert it into the standard $|\Psi^-\rangle$ state using local operations by Eq. (5). Because of the symmetry property any two parties can join together to help the other two get a $|\Psi^-\rangle$. Note that the unlockability property implies the state must not be fully separable, or no entanglement could be distilled between separated parties, even when some of the parties come together. Because the state is both non-distillable and entangled, it is by definition a *bound-entangled state* [6,7].

Now we consider the mixed state of five parties A, B,

C, D, and E

$$M = \rho^{ACBD} \otimes \rho^{ABCE} \tag{8}$$

where ρ^{ACBD} (call it state 1) and ρ^{ABCE} (call it state 2) are the states of Eq. (7) but with the qubits assigned to different parties. Thus parties A, B, C and D each have a one qubit subsystem of state ρ^{ACBD} and similarly parties A, B, C and E each have a one qubit subsystem of state ρ^{ABCE}. Thus the parties A, B, C, D and E have Hilbert spaces of size 4, 4, 4, 2, and 2. Technically ρ^{ACBD} could be written as ρ^{ABCD} due to the symmetry property but it will be useful to have it explicitly written in the form where it is an unknown Bell state shared between A and C, and the same state shared by B and D. The state M is illustrated in Figure 1a. M is the tensor product of two density matrices, neither of which is independently distillable. We now show how to distill a $|\Psi^-\rangle$ between D and E.

In the distillation procedure A and B use state 1 to "teleport" state 2 to C and D. First, party A teleports her half of the unknown Bell state she shares with B (which is part of state 2 and shown by the solid arrow connecting A and B in Figure 1a and part of state 2) to C using the unknown Bell state she shares with C (which is part of state 1, shown by the dashed arrow connecting A and C in the figure). This results in the situation of Figure 1b, where now C shares an unknown Bell state with B, her half of which has additionally picked up the unknown rotation σ_i from having been teleported with an incorrect Bell state $|\Psi_i\rangle$. The Bell state connecting A and C is gone in the figure, since it has been expended performing the teleportation. Then B teleports his half of that state to D using the unknown Bell state (again $|\Psi_i\rangle$ that they share, resulting in the situation of Figure 1c, where now C and D share the unknown Bell state originally shared by A and B, both halves of which having been rotated by σ_i. It is important to note here that because of the structure of ρ^{ACBD} this is the *same* σ_i. Now, using Eq. (6) and the fact that σ_i^2 is the identity (once again except for a phase), we can see that the σ_i's cancel and we are left with the state ρ^{CDCE}. This is the same form as the four-party unlockable state (Eq. (7)) but with one party sharing two of the qubits, and it is therefore distillable into a pure EPR pair shared by D and E.

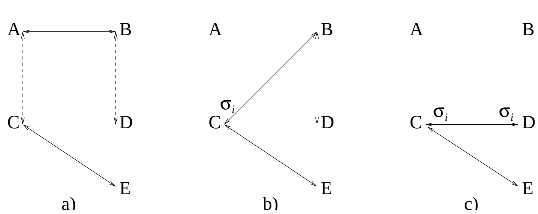

FIG. 1. How to distill the state M into an EPR pair between D and E: a) The state M, with the two identical but unknown Bell states of ρ^{ABCD} shown as dashed arrows, and those of ρ^{ABCE} as solid arrows. b) A has teleported her half of the unknown Bell state she shares with B to C, using the unknown Bell state $|\Psi_i\rangle$ she shares with C. The state has picked up a factor of σ_i. c) B has teleported his half of the unknown Bell state he shared originally with A and now shares with C to D using his unknown Bell state shared with D (again, $|\Psi_i\rangle$). The state has picked up another factor of σ_i. The σ_i's cancel each other and the final state of CDE is of the form of Eq. (7), but with party C having two of the qubits *i.e.* ρ^{CCDE}. This is the unlockable bound-entangled state of [14] in its "unlocked" configuration, and can therefore be distilled into a DE EPR pair by C simply measuring which Bell state she has and telling D and E which one they have since the two are the same.

M cannot be distilled into EPR pairs between any of the other parties. This is because if we give the five parties the additional power of having D and E in the same room, then M is just two copies of ρ^{ABCD} which are known not to be distillable (by definition if ρ is not distillable, then neither is $\rho^{\otimes N}$). To construct a state out of tensor products of bound-entangled states that is distillable into any kind of pure entanglement, it is sufficient to symmetrize M, *i.e.*

$$M_{\mathrm{S}} = \rho^{ABCD} \otimes \rho^{ABCE} \otimes \rho^{ABDE} \otimes \rho^{ACDE} \otimes \rho^{BCDE} \,. \tag{9}$$

Then the distillation protocol just described can be used to obtain an EPR pair between any two of the parties, and using more copies of M_{S} one can obtain EPR pairs between all pairs of parties. Once this is accomplished any arbitrary multi-party entangled state can be constructed by one party creating it in his lab and teleporting the pieces as needed to the others.

Because M (Eq. (8)) is distillable, it cannot be that the original state ρ^{ABCD} is asymptotically unentangled. If it were, then many copies N of ρ^{ABCD} and ρ^{ABCE} could be created arbitrarily precisely using a number of EPR pairs sublinear in N. These could be used to create N copies of M which could then be distilled into N pure EPR pairs between D and E. These DE EPR pairs would, to arbitrarily high probability, pass any test that pure EPR pairs would pass. Thus, an amount of entanglement sublinear in N would have been converted into N EPR pairs by LOCC, which is impossible [5].

In fact, all unlockable bound-entangled states are asymptotically inseparable. This is because when some subset S of the parties possessing such a state come together in the same lab the state becomes distillable. If the state were asymptotically unentangled then it could be made arbitrarily precisely with asymptotically no entanglement even when parties in S are actually together in the same lab (it cannot hurt for them to be together as they can conveniently ignore this fact as they carry

out whatever procedure results in the creation of the state). But then they can distill a finite amount of arbitrarily pure entanglement per state from the sublinear amount of entanglement they started with, which is impossible. It is worth noting then that the unlockable bound-entangled states are the first states shown to be true bound-entangled states in the sense of both being non-distillable and being non-separable asymptotically.

It is clearly a necessary condition for superactivation that at least one of the states involved must not be asymptotically unentangled. It is by no means a sufficient one, however, since the states ρ^{ABCD} and ρ^{EFGH} are each not asymptotically unentangled but $\rho^{ABCD} \otimes \rho^{EFGH}$ is not distillable as the two pieces are on disconnected sets of parties.

In the individual states ρ^{ABCD} and ρ^{ABCE}, every party is separated from every other party by at least one separable cut. In order for the combined state M to be distillable into a DE EPR pair, and for $M_{\rm S}$ to be distillable into EPR pairs between any pair of parties, it is necessary that the parties who get EPR pairs no longer be separated by any separable cut, as is indeed the case by construction for these states. Using this observation, Dür has reported a whole family of superactivated states [16] based on the unlockable bound-entangled states of [11–13]. References [11,12] also discuss separable cuts and their relation to distillibility in more detail.

In conclusion, we have shown that asymptotically entangled states exist from which no pure entanglement can be distilled. This has been suspected for some time but ours is the first such example for which it has been proved. Further we have shown the surprising fact that distillable entanglement is not additive by showing two undistillable asymptotically entangled states that when combined gives a distillable state.

Many future directions are suggested by this work. Here we have shown four party example of a state that is asymptotically entangled but not distillable. An interesting question is whether such a state can be found for two parties. Since the first writing of this letter, this question has been answered in the affirmative in [9]. Another direction for further research is to find bipartite states that show the non-additivity of distillable entanglement. Such examples have been shown to exist if the NPT-boud entangled states are truly bound entangled [17].

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